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A note on nonlinear $\mathcal{H}_\infty$ control of two-block interconnected systems

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Abstract

In this paper nonlinear $\mathcal{H}_\infty$ control for a class of two-block interconnected systems is investigated. The situation where the *regular* nonlinear $\mathcal{H}_\infty$ suboptimal control problem is solvable for one of the blocks is considered. An auxiliary nonlinear system is defined from the original two-block system, and it is shown that a feedback solution to the nonlinear $\mathcal{H}_\infty$ suboptimal control problem for this auxiliary system, also applies to the nonlinear $\mathcal{H}_\infty$ suboptimal control problem for the original two-block system. The advantage of the auxiliary problem to the original problem is that the auxiliary penalty variable has lower dimension than the original penalty variable.

1. Introduction

In [1] decomposition ideas from linear theory were used to solve the (singular) *nonlinear* $\mathcal{H}_\infty$ problem. Also in [2] a decomposition idea was used. In this paper we will show that the results obtained in [3] for the *totally singular* case, can be extended to a more general situation.

The system under consideration is an affine nonlinear system, which we will denote by Σ, given by state space equations of the form

$$\Sigma: \quad \begin{cases} \dot{x} = f(x) + g(x)u + k(x)d, & f(x_0) = 0 \\ z = h(x,u), \quad h(x_0, 0) = 0 \end{cases} \tag{1}$$

where $x = [x^1, \ldots, x^n]^T$ are local coordinates for an n-dimensional state space manifold $\mathcal{X}$, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^r$ the disturbance input and $z \in \mathbb{R}^s$ the penalty variable. f, g, k and h are all C^k with $k \geq 2$. We will assume that there exist local coordinates $x = [x^1, \ldots x^q, x^{q+1}, \ldots, x^n]^T$, $1 \leq q < n$, such that f, g, k and h take the form

$$f(x) = \begin{bmatrix} f_1(x_1) + g_1(x_1)z_2 \\ f_2(x_1, x_2) \end{bmatrix}, \; g(x) = \begin{bmatrix} 0 \\ g_2(x_1, x_2) \end{bmatrix}$$

$$k(x) = \begin{bmatrix} k_1(x_1) \\ k_2(x_1, x_2) \end{bmatrix}, \quad h(x,u) = \begin{bmatrix} h_1(x_1) \\ h_2(x_1, x_2) \\ h_3(x_1, x_2, u) \end{bmatrix}$$

for $x_1 = [x^1, \ldots x^q]^T$ and $x_2 = [x^{q+1}, \ldots, x^n]^T$. Moreover, $z_1 \in \mathbb{R}^{s_1}$, $z_2 \in \mathbb{R}^{s_2}$, and $z_3 \in \mathbb{R}^{s_3}$, where $s_1 \geq 1$ and $s = s_1 + s_2 + s_3$. Thus, Σ can be viewed as a two-block interconnected system where the system blocks are two general nonlinear systems: a system Σ_1 where x_1 is the state, z_1 is the output and (z_2, d) are the inputs and a system Σ_2 where x_2 is the state, z_2 is the output and (u, x_1, d) are the inputs.

We will now state the problem that we want to consider in this paper.

Problem 1.1 *Let γ be a fixed positive constant. Solve the state feedback nonlinear $\mathcal{H}_\infty$ suboptimal control problem associated with the system Σ, i.e. find a nonlinear static state feedback*

$$u = \alpha(x), \quad \alpha(x_0) = 0 \tag{2}$$

such that the closed loop system (1), (2) has L_2-gain $\leq \gamma$ from d to z.

Now, consider the system Σ and define the pre-Hamiltonian $K_\gamma : T^*\mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}$ as

$$K_\gamma(x,p,u,d) = p^T(f(x) + g(x)u + k(x)d) - \frac{1}{2}\gamma^2 d^T d + \frac{1}{2} z^T z \tag{3}$$

where $x = [x^1, \ldots, x^n]^T$, $p = [p^1, \ldots, p^n]^T$ are natural coordinates for the cotangent bundle $T^*\mathcal{X}$. For the sake of completeness, we will write down the following well-known result (cf. [4]).

Proposition 1.1 *Consider the closed loop system (1), (2). Suppose that there exists a non-negative C^1 solution $V : \mathcal{X} \to \mathbb{R}^+$ to the differential dissipation inequality*

$$K_\gamma(x, V_x^T(x), \alpha(x), d) \leq 0, \quad V(x_0) = 0, \quad \textit{for all } d \in \mathbb{R}^r \tag{4}$$

Then the state feedback (2) solves Problem 1.1. Moreover, if the closed loop system (1), (2) is zero-state observable with x_0 as the zero-state (i.e. $z(t) \equiv 0$, $d(t) \equiv 0$ implies that $x(t) \equiv x_0$), then $V(x) > 0$ for all $x \neq x_0$, and x_0 is a locally asymptotically stable equilibrium when $d(t) \equiv 0$. If $V(x)$ is proper, then x_0 is a globally asymptotically stable equilibrium when $d(t) \equiv 0$.

2. Main Results

For the derivation of our main results in this section the following assumption will be instrumental.

Assumption 2.1 *Consider the two-block interconnected system Σ given by (1). Let $\mathcal{X}_1$ be the submanifold of $\mathcal{X}$ with local coordinates $[x^1, \ldots, x^q]^T$. Then there exists a non-negative C^v ($k \geq v \geq 2$) solution $P : \mathcal{X}_1 \to \mathbb{R}^+$ to the Hamilton-Jacobi inequality*

$$\begin{aligned} &P_{x_1}(x_1) f_1(x_1) + \\ &\frac{1}{2} P_{x_1}(x_1) \left[\frac{1}{\gamma^2} k_1(x_1) k_1^T(x_1) - g_1(x_1) g_1^T(x_1) \right] P_{x_1}^T(x_1) \\ &\qquad + \frac{1}{2} h_1^T(x_1) h_1(x_1) \leq 0, \; P(x_{1,0}) = 0 \end{aligned} \tag{5}$$

where $x_0 = [x_{1,0}^T, x_{2,0}^T]^T$.

In consequence of Assumption 2.1, the *regular* nonlinear $\mathcal{H}_\infty$ suboptimal control problem associated with the subsystem

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)z_2 + k_1(x_1)d, \;\; f_1(x_{1,0}) = 0 \\ z &= \begin{bmatrix} h_1(x_1) \\ z_2 \end{bmatrix}, \;\; h_1(x_{1,0}) = 0 \end{aligned} \tag{6}$$

is solvable when z_2 is regarded as the control input. The C^{v-1} state feedback solution is then given by [4]

$$z_2^* = -g_1^T(x_1)P_{x_1}^T(x_1) \tag{7}$$

Motivated by the results in [1], we will now use this fact to define an auxiliary nonlinear system from the original two-block interconnected system Σ. The auxiliary nonlinear system, which we will denote by $\bar{\Sigma}$, is again an affine nonlinear system given by state space equations of the form

$$\bar{\Sigma}: \quad \left\{ \begin{array}{l} \dot{x} = \bar{f}(x) + g(x)u + k(x)\bar{d}, \;\; \bar{f}(x_0) = 0 \\ \bar{z} = \bar{h}(x,u), \;\; \bar{h}(x_0, 0) = 0 \end{array} \right. \tag{8}$$

where

$$\bar{d} = d - d^*, \; d^* = \frac{1}{\gamma^2}k_1^T(x_1)P_{x_1}^T(x_1) \tag{9}$$

$$\bar{f}(x) = \begin{bmatrix} f_1(x_1) + g_1(x_1)z_2 + k_1(x_1)d^* \\ f_2(x_1,x_2) + k_2(x_1,x_2)d^* \end{bmatrix} \tag{10}$$

$$\bar{h}(x,u) = \begin{bmatrix} h_2(x_1,x_2) + g_1^T(x_1)P_{x_1}^T(x_1) \\ h_3(x_1,x_2,u) \end{bmatrix} \tag{11}$$

Using the Hamilton-Jacobi inequality (5) the following lemma can be easily proven.

Lemma 2.1 *Let* $z \in \mathbb{R}^s$ *and* $\bar{z} \in \mathbb{R}^{s-s_1}$ *denote the penalty variable of* Σ *and* $\bar{\Sigma}$ *respectively. Then the following inequality is satisfied*

$$\frac{1}{2}\bar{z}^T\bar{z} \geq P_{x_1}(x_1)(f_1(x_1) + g_1(x_1)z_2) + \frac{1}{2\gamma^2}||k_1^T(x_1)P_{x_1}^T(x_1)||^2 + \frac{1}{2}z^Tz \tag{12}$$

Our main result relies on the next lemma.

Lemma 2.2 *Let* K_γ *and* $\bar{K}_\gamma$ *denote the pre-Hamiltonian of* Σ *and* $\bar{\Sigma}$ *respectively. Then it follows that*

$$K_\gamma(x, p + P_x^T(x_1), u, d) \leq \bar{K}_\gamma(x,p,u,\bar{d}) \tag{13}$$

Proof: Using Lemma 2.1 and the fact that $\bar{f}(x) + k(x)\bar{d} = f(x) + k(x)d$ it follows that

$$\begin{aligned} &\bar{K}_\gamma(x,p,u,\bar{d}) \geq p^T(f(x) + g(x)u + k(x)d) \\ &+ P_{x_1}(x_1)(f_1(x_1) + g_1(x_1)z_2) + \frac{1}{2\gamma^2}||k_1^T(x_1)P_{x_1}^T(x_1)||^2 \\ &\qquad + \frac{1}{2}z^Tz - \frac{1}{2}\gamma^2\bar{d}^T\bar{d} \end{aligned} \tag{14}$$

Moreover, from (9) it follows that

$$\begin{aligned} \frac{1}{2}\gamma^2\bar{d}^T\bar{d} &= \frac{1}{2}\gamma^2 d^Td + \frac{1}{2\gamma^2}||k_1^T(x_1)P_{x_1}^T(x_1)||^2 \\ &\quad - P_{x_1}(x_1)k_1(x_1)d \end{aligned} \tag{15}$$

Inserting (15) in (14) then gives (13) since $P_{x_2}(x_1) = 0$. ∎

We are now ready to state our main result which is an immediate consequence of the last lemma.

Theorem 2.1 *Suppose that the static state feedback*

$$u = \alpha(x), \quad \alpha(x_0) = 0 \tag{16}$$

solves the nonlinear $\mathcal{H}_\infty$ *suboptimal control problem for the auxiliary system* $\bar{\Sigma}$ *in the sense that there exists a non-negative* C^1 *solution* $W : \mathcal{X} \to \mathbb{R}^+$ *to the differential dissipation inequality*

$$\bar{K}_\gamma(x, W_x^T(x), \alpha(x), \bar{d}) \leq 0, \; W(x_0) = 0, \; \textit{for all } \bar{d} \in \mathbb{R}^r \tag{17}$$

Then the state feedback (16) also solves Problem 1.1 and the differential dissipation inequality

$$K_\gamma(x, V_x^T(x), \alpha(x), d) \leq 0, \;\; V(x_0) = 0, \;\; \textit{for all } d \in \mathbb{R}^r \tag{18}$$

holds for the non-negative C^1 *function* $V : \mathcal{X} \to \mathbb{R}^+$ *given by* $V = W + P$.

Remark 2.1 *It is easy to see that systems that admit the following decomposition*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)z_2 + k_1(x_1)d \\ \dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)z_3 + k_2(x_1,x_2)d \\ &\vdots \\ \dot{x}_{p-1} &= f_{p-1}(x_1,\dots,x_{p-1}) \\ &\qquad + g_{p-1}(x_1,\dots,x_{p-1})z_{p-1} \\ &\qquad\quad + k_{p-1}(x_1,\dots,x_{p-1})d \\ \dot{x}_p &= f_p(x_1,\dots,x_p) \\ &\qquad + g_p(x_1,\dots,x_p)u \\ &\qquad\quad + k_p(x_1,\dots,x_p)d \\ z &= \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \\ z_{p+1} \end{bmatrix} = \begin{bmatrix} h_1(x_1) \\ h_2(x_1,x_2) \\ \vdots \\ h_p(x_1,\dots,x_p) \\ h_{p+1}(x_1,\dots,x_p,u) \end{bmatrix} \end{aligned} \tag{19}$$

are candidates for recursive use of the results in Theorem 2.1.

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